

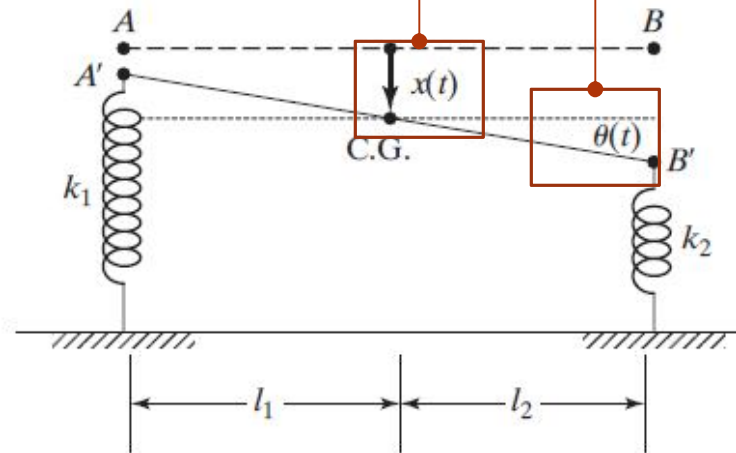
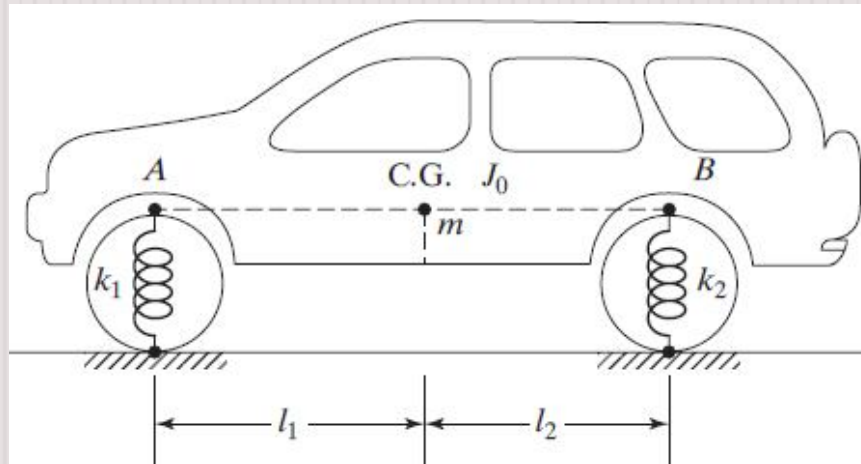
Mechanical Vibrations

2DoFs System

Two degrees of freedom system

- Examples

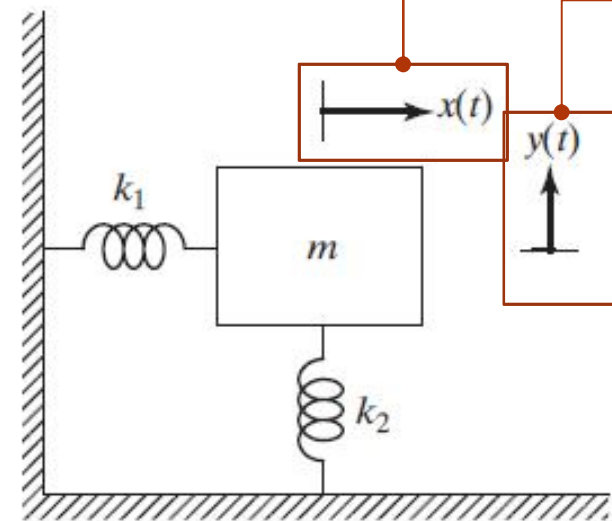
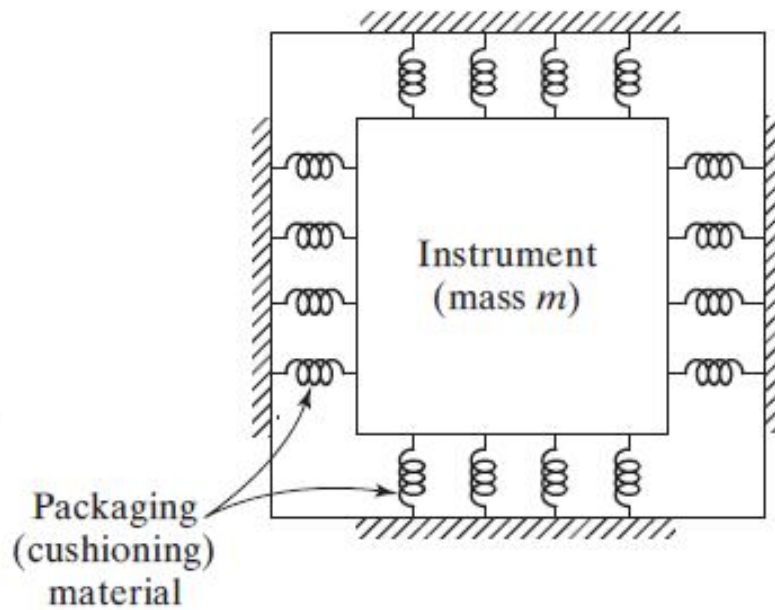
2 DoF : $x(t)$ and $\theta(t)$



Two degrees of freedom system

Examples

DoF : $x(t)$ and $y(t)$



Free Vibration

- General physical system:
- Consider the masses springs system shown:

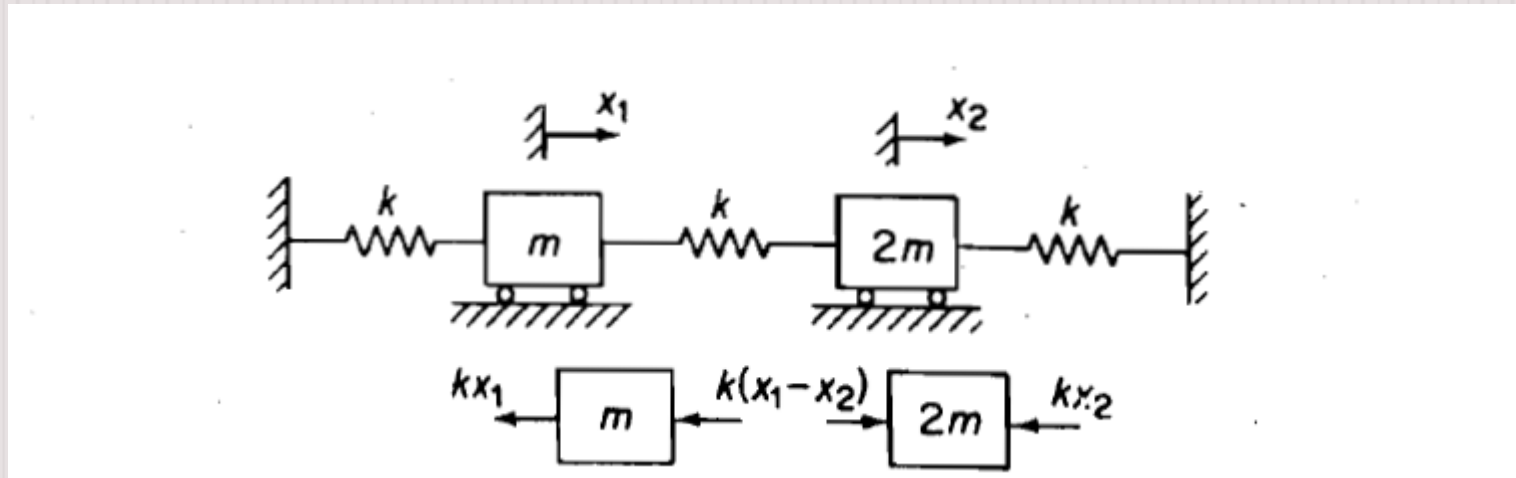


Figure (1)

The equations of motion can be obtained using Newton's second law as follows:

Free Vibration

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) \\ 2m\ddot{x}_2 &= -k(x_2 - x_1) - kx_2 \end{aligned} \quad (5.1.1)$$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$2m\ddot{x}_2 + 2kx_2 - kx_1 = 0$$

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

$$\mathbf{M} \{ \ddot{\mathbf{x}} \} + \mathbf{K} \{ \mathbf{x} \} = 0$$

Free Vibration

For the normal mode of oscillation, each mass undergoes harmonic motion of the same frequency, passing through the equilibrium position simultaneously. For such motion, we can let

$$\begin{aligned}x_1 &= A_1 \sin \omega t \quad \text{or} \quad A_1 e^{i\omega t} \\x_2 &= A_2 \sin \omega t \quad \text{or} \quad A_2 e^{i\omega t}\end{aligned}\tag{5.1.2}$$

Substituting these into the differential equations, we have

$$-\omega^2 \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \sin(\omega t) + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \sin(\omega t) = 0$$

$$\left(\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \right) \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \sin(\omega t) = 0$$

$$\left(\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \right) \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0$$

$$(\mathbf{K} - \omega^2 \mathbf{M}) \{\mathbf{A}\} = 0$$

Free Vibration

$$\begin{bmatrix} (2k - \omega^2 m) & -k \\ -k & (2k - 2\omega^2 m) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5.1.4)$$

This equation is satisfied for any A_1 and A_2 if the determinant of the above equations is zero.

$$\begin{vmatrix} (2k - \omega^2 m) & -k \\ -k & (2k - 2\omega^2 m) \end{vmatrix} = 0 \quad (5.1.5)$$

Letting $\omega^2 = \lambda$ and multiplying out, the foregoing determinant results in a second-degree algebraic equation that is called the *characteristic equation*.

$$\lambda^2 - \left(3 \frac{k}{m}\right) \lambda + \frac{3}{2} \left(\frac{k}{m}\right)^2 = 0 \quad (5.1.6)$$

The two roots λ_1 and λ_2 of this equation are the *eigenvalues* of the system:

$$\begin{aligned} \lambda_1 &= \left(\frac{3}{2} - \frac{1}{2} \sqrt{3} \right) \frac{k}{m} = 0.634 \frac{k}{m} \\ \lambda_2 &= \left(\frac{3}{2} + \frac{1}{2} \sqrt{3} \right) \frac{k}{m} = 2.366 \frac{k}{m} \end{aligned} \quad (5.1.7)$$

Free Vibration

and the *natural frequencies* of the system are

$$\omega_1 = \lambda_1^{1/2} = \sqrt{0.634 \frac{k}{m}}$$

$$\omega_2 = \lambda_2^{1/2} = \sqrt{2.366 \frac{k}{m}}$$

From Eq. (5.1.3), two expressions for the ratio of the amplitudes are found:

$$\frac{A_1}{A_2} = \frac{k}{2k - \omega^2 m} = \frac{2k - 2\omega^2 m}{k} \quad (5.1.8)$$

Substitution of the natural frequencies in either of these equations leads to the ratio of the amplitudes. For $\omega_1^2 = 0.634k/m$, we obtain

$$\left(\frac{A_1}{A_2} \right)^{(1)} = \frac{k}{2k - \omega_1^2 m} = \frac{1}{2 - 0.634} = 0.731$$

which is the amplitude ratio corresponding to the first natural frequency.

Similarly, using $\omega_2^2 = 2.366 k/m$, we obtain

$$\left(\frac{A_1}{A_2} \right)^{(2)} = \frac{k}{2k - \omega_2^2 m} = \frac{1}{2 - 2.366} = -2.73$$

for the amplitude ratio corresponding to the second natural frequency. Equation (5.1.8) enables us to find only the ratio of the amplitudes and not their absolute values, which are arbitrary.

If one of the amplitudes is chosen equal to 1 or any other number, we say that the amplitude ratio is *normalized* to that number. The normalized amplitude ratio is then called the *normal mode* and is designated by $\phi_i(x)$.

Free Vibration

The two normal modes of this example, which we can now call *eigenvectors*, are

$$\phi_1(x) = \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix} \quad \phi_2(x) = \begin{Bmatrix} -2.73 \\ 1.00 \end{Bmatrix}$$

Each normal mode oscillation can then be written as

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{(1)} = A_1 \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix} \sin(\omega_1 t + \psi_1)$$

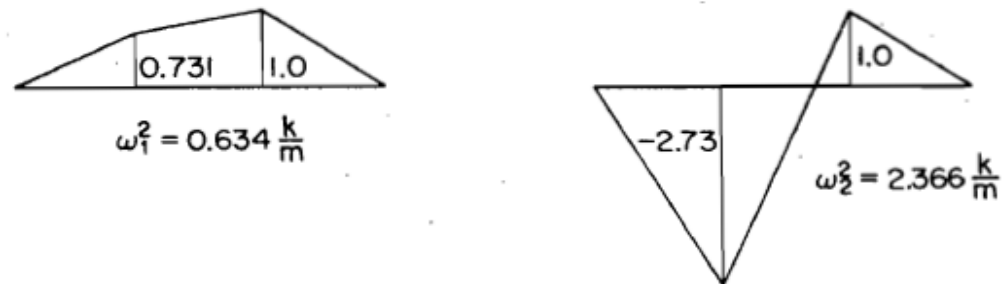
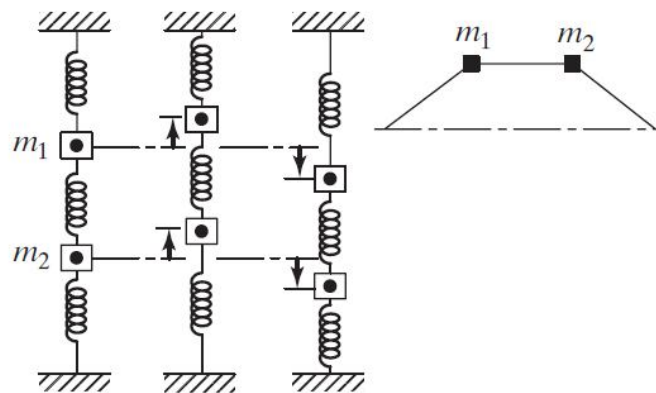
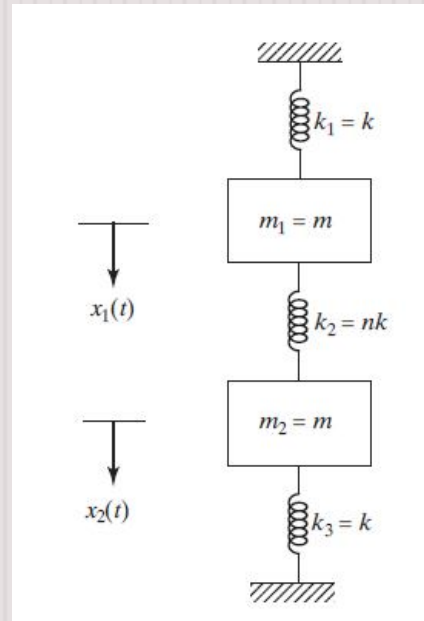


FIGURE 5.1.2. Normal modes of the system shown in Figure 5.1.1.

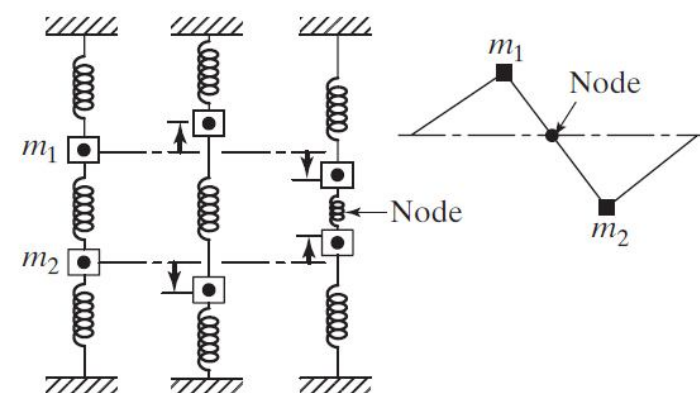
$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{(2)} = A_2 \begin{Bmatrix} -2.73 \\ 1.00 \end{Bmatrix} \sin(\omega_2 t + \psi_2)$$

These normal modes are displayed graphically in Fig. 5.1.2. In the first normal mode, the two masses move in phase; in the second mode, the two masses move in opposition, or out of phase with each other.

Free Vibration



(a) First mode



(b) Second mode

Free Vibration

We now describe the rotational system shown in Fig. 5.1.3 with coordinates θ_1 and θ_2 measured from the inertial reference. From the free-body diagram of two disks, the torque equations are

$$\begin{aligned} J_1 \ddot{\theta}_1 &= -K_1 \theta_1 + K_2 (\theta_2 - \theta_1) \\ J_2 \ddot{\theta}_2 &= -K_2 (\theta_2 - \theta_1) - K_3 \theta_2 \end{aligned} \quad (5.1.9)$$

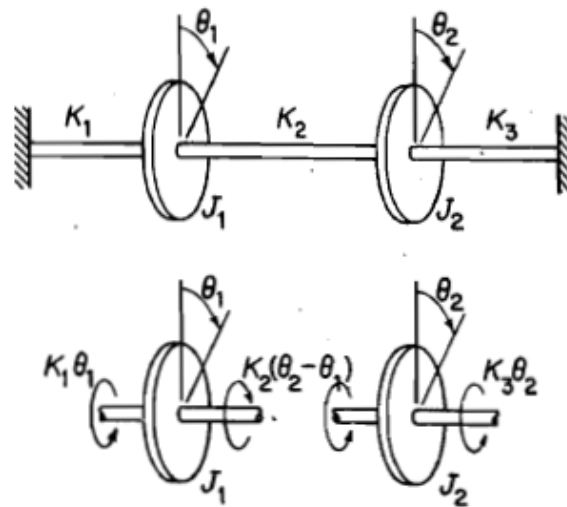


FIGURE 5.1.3.

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} (K_1 + K_2) & -K_2 \\ -K_2 & (K_2 + K_3) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (5.1.10)$$

Free Vibration

The matrix

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

is known as the mass matrix and the matrix

$$\begin{bmatrix} (K_1 + K_2) & -K_2 \\ -K_2 & (K_2 + K_3) \end{bmatrix}$$

is known as the stiffness matrix. Both of these matrices will be discussed in detail in Chapter 6.

A few points of interest should be noted. The stiffness matrix is symmetric about the diagonal and the mass matrix is diagonal. Thus, the square matrices are equal to their transpose, i.e., $[k]^T = [k]$, and $[m]^T = [m]$. In addition, for the discrete mass system with coordinates chosen at each mass, the mass matrix is diagonal and its inverse is simply the inverse of each diagonal element, i.e., $[m]^{-1} = [1/m]$.

Free Vibration

EXAMPLE 5.1.3 Coupled Pendulum

In Fig. 5.1.4 the two pendulums are coupled by means of a weak spring k , which is unstrained when the two pendulum rods are in the vertical position. Determine the normal mode vibrations.

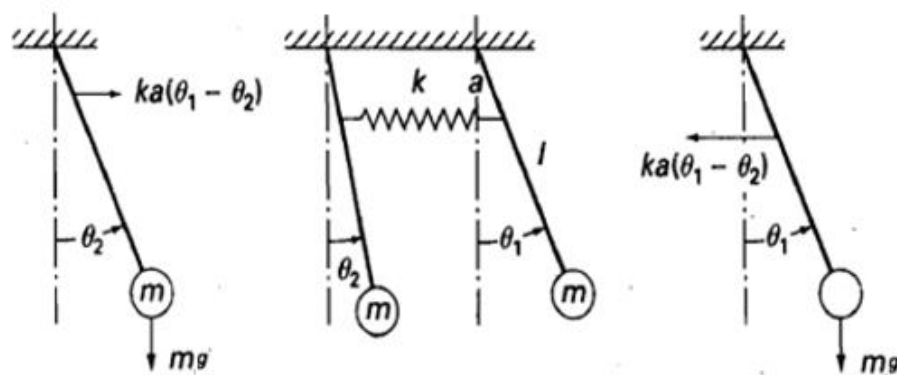


FIGURE 5.1.4. Coupled pendulum.

Solution Assuming the counterclockwise angular displacements to be positive and taking moments about the points of suspension, we obtain the following equations of motion for small oscillations

$$ml^2\ddot{\theta}_1 = -mgl\theta_1 - ka^2(\theta_1 - \theta_2)$$

$$ml^2\ddot{\theta}_2 = -mgl\theta_2 + ka^2(\theta_1 - \theta_2)$$

Free Vibration

which in matrix notation becomes

$$ml^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} (ka^2 + mgl) & -ka^2 \\ -ka^2 & (ka^2 + mgl) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (5.1.10)$$

Assuming the normal mode solutions as

$$\theta_1 = A_1 \cos \omega t \quad \text{or} \quad A_1 e^{i\omega t}$$

$$\theta_2 = A_2 \cos \omega t \quad \text{or} \quad A_2 e^{i\omega t}$$

Free Vibration

the natural frequencies and mode shapes are

$$\omega_1 = \sqrt{\frac{g}{l}} \quad \omega_2 = \sqrt{\frac{g}{l} + 2\frac{k}{m}\frac{a^2}{l^2}}$$

$$\left(\frac{A_1}{A_2}\right)^{(1)} = 1.0 \quad \left(\frac{A_1}{A_2}\right)^{(2)} = -1.0$$

Thus, in the first mode, the two pendulums move in phase and the spring remains unstretched. In the second mode, the two pendulums move in opposition and the coupling spring is actively involved with a node at its midpoint. Consequently, the natural frequency is higher.

Free Vibration

EXAMPLE 5.3.2

Determine the normal modes of vibration of an automobile simulated by the simplified 2-DOF system with the following numerical values (see Fig. 5.3.5):

$$W = 3220 \text{ lb} \quad l_1 = 4.5 \text{ ft} \quad k_1 = 2400 \text{ lb/ft}$$

$$J_c = \frac{W}{g} r^2 \quad l_2 = 5.5 \text{ ft} \quad k_2 = 2600 \text{ lb/ft}$$

$$r = 4 \text{ ft} \quad l = 10 \text{ ft}$$

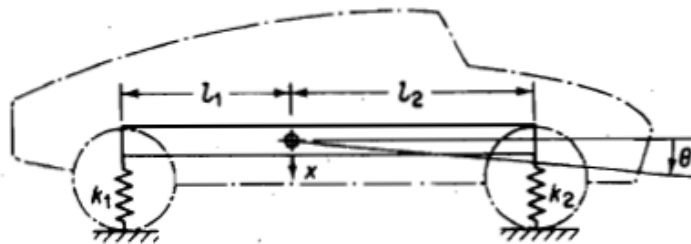
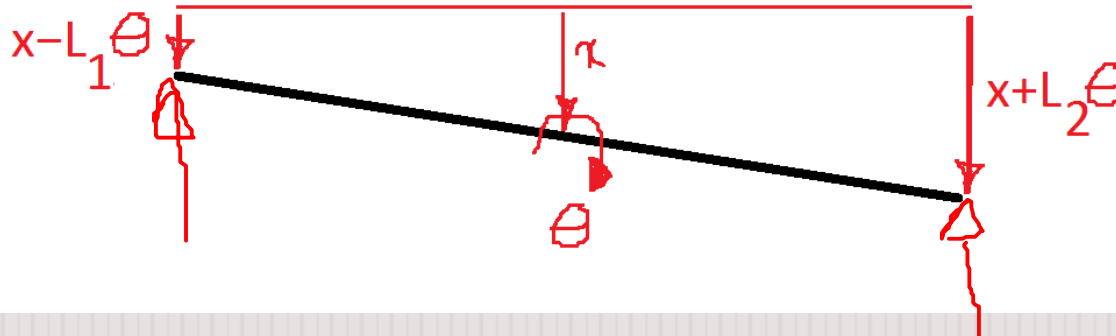


FIGURE 5.3.5.



$$\sum F_x = M\ddot{x}$$

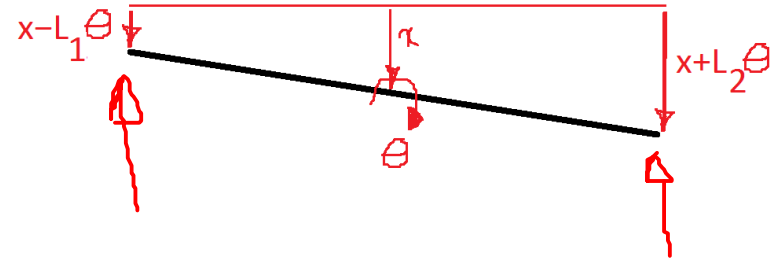
$$-k_1(x - L_1\theta) - k_2(x + L_2\theta) = M\ddot{x}$$

$$M\ddot{x} + (k_1 + k_2)x + (k_2L_2 - k_1L_1)\theta = 0$$

$$\sum M_G = I_G\ddot{\theta}$$

$$k_1(x - L_1\theta)L_1 - k_2(x + L_2\theta)L_2 = I_G\ddot{\theta}$$

$$I_G\ddot{\theta} + (k_1L_1^2 + k_2L_2^2)\theta + (k_2L_2 - k_1L_1)x = 0$$



$$M\ddot{x} + (k_1 + k_2)x + (k_2L_2 - k_1L_1)\theta = 0$$

$$I_G\ddot{\theta} + (k_1L_1^2 + k_2L_2^2)\theta + (k_2L_2 - k_1L_1)x = 0$$

$$\begin{bmatrix} M & 0 \\ 0 & I_G \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & (k_2L_2 - k_1L_1) \\ (k_2L_2 - k_1L_1) & (k_1L_1^2 + k_2L_2^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} (k_1 + k_2) - M\omega^2 & (k_2L_2 - k_1L_1) \\ (k_2L_2 - k_1L_1) & (k_1L_1^2 + k_2L_2^2) - I_G\omega^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} (k_1 + k_2) - M\omega^2 & (k_2 L_2 - k_1 L_1) \\ (k_2 L_2 - k_1 L_1) & (k_1 L_1^2 + k_2 L_2^2) - I_G \omega^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} (k_1 + k_2) - M\omega^2 & (k_2 L_2 - k_1 L_1) \\ (k_2 L_2 - k_1 L_1) & (k_1 L_1^2 + k_2 L_2^2) - I_G \omega^2 \end{vmatrix} = 0 \rightarrow \omega_{n1}, \omega_{n2}$$

Mode - shape :

$$\left((k_1 + k_2) - M\omega^2 \right) x + (k_2 L_2 - k_1 L_1) \theta = 0 \rightarrow \frac{x}{\theta} = \frac{(k_1 L_1 - k_2 L_2)}{\left((k_1 + k_2) - M\omega^2 \right)}$$

$$\psi_1 = \left\{ \begin{array}{c} \frac{(k_1 L_1 - k_2 L_2)}{\left((k_1 + k_2) - M\omega_{n1}^2 \right)} \\ 1 \end{array} \right\} \quad \psi_2 = \left\{ \begin{array}{c} \frac{(k_1 L_1 - k_2 L_2)}{\left((k_1 + k_2) - M\omega_{n2}^2 \right)} \\ 1 \end{array} \right\}$$

$$\omega_1 = 6.90 \text{ rad/s} = 1.10 \text{ cps}$$

$$\omega_2 = 9.06 \text{ rad/s} = 1.44 \text{ cps}$$

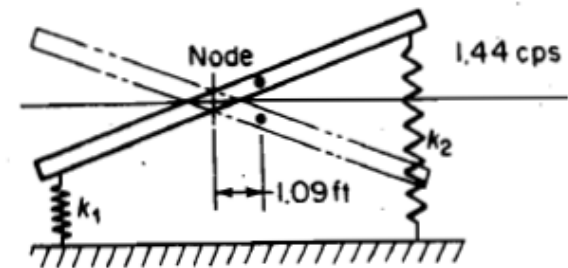
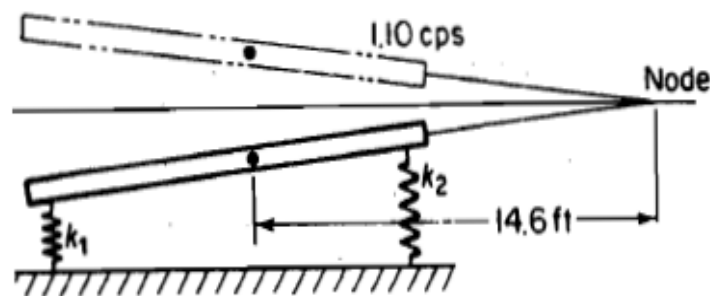
The amplitude ratios for the two frequencies are

$$\left(\frac{x}{\theta}\right)_{\omega_1} = -14.6 \text{ ft/rad} = -3.06 \text{ in./deg}$$

$$\left(\frac{x}{\theta}\right)_{\omega_2} = 1.09 \text{ ft/rad} = 0.288 \text{ in./deg}$$

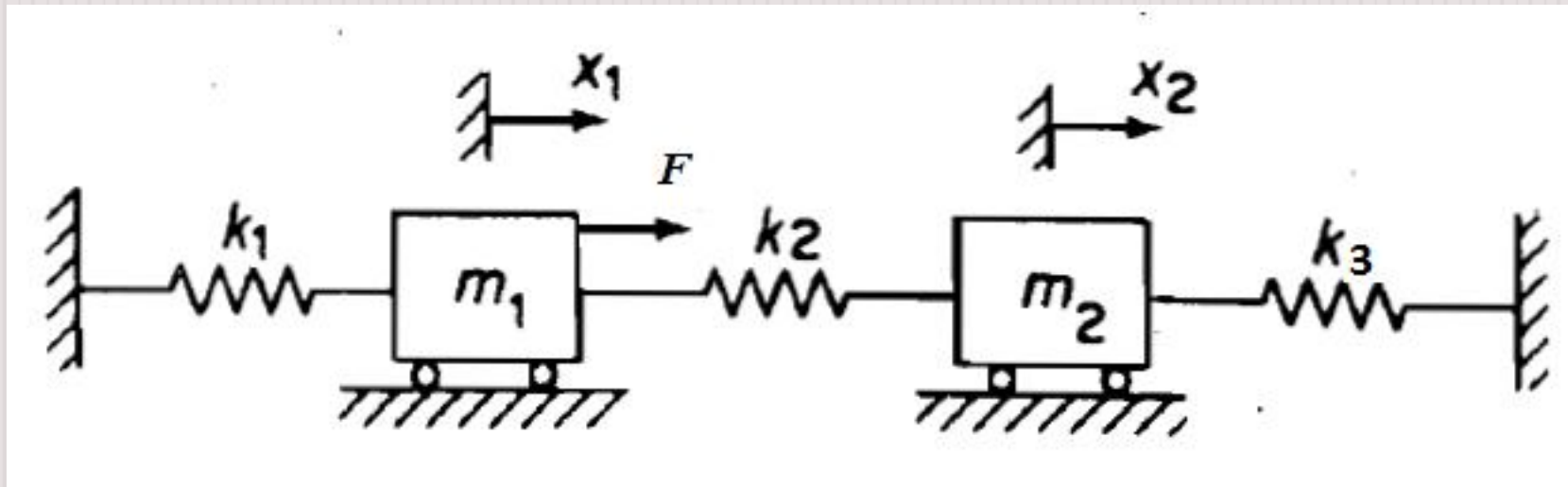
The mode shapes are illustrated by the diagrams of Fig. 5.3.6.

In interpreting these results, the first mode, $\omega_1 = 6.9 \text{ rad/s}$ is largely vertical translation very small rotation, whereas the second mode, $\omega_2 = 9.06 \text{ rad/s}$ is mostly rotation. This suggests that we could have made a rough approximation for these modes as two 1-DOF systems.



Forced Vibration

- General physical system:
- Consider the masses springs system shown:



$$F = F_1 \sin(\omega t)$$

Forced Vibration

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \sin \omega t$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin \omega t$$

$$-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t) + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t) = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \sin(\omega t)$$

$$\begin{bmatrix} k_{11} - m_1 \omega^2 & k_{12} \\ k_{21} & k_{22} - m_2 \omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t) = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \sin(\omega t)$$

Forced Vibration

$$\begin{bmatrix} (k_{11} - m_1\omega^2) & k_{12} \\ k_{21} & (k_{22} - m_2\omega^2) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

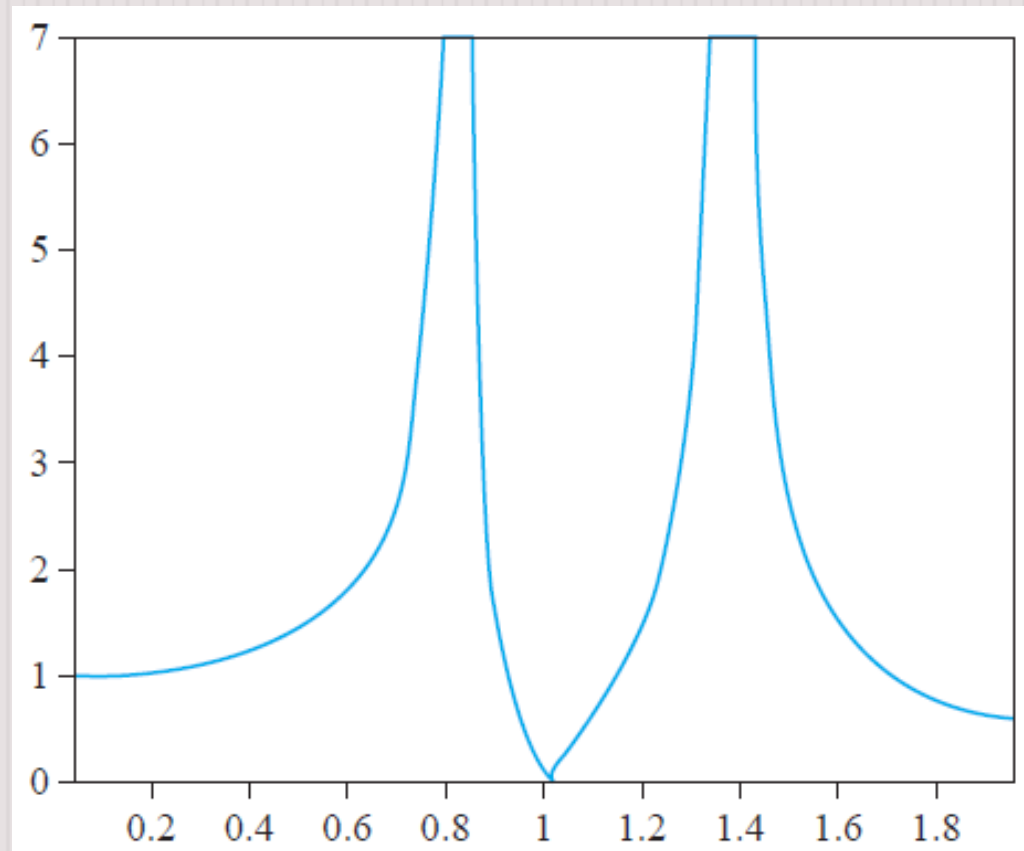
$$[Z(\omega)] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = [Z(\omega)]^{-1} \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} = \frac{\text{adj } [Z(\omega)] \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}}{|Z(\omega)|}$$

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{1}{|Z(\omega)|} \begin{bmatrix} (k_{22} - m_2\omega^2) & -k_{12} \\ -k_{21} & (k_{11} - m_1\omega^2) \end{bmatrix} \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

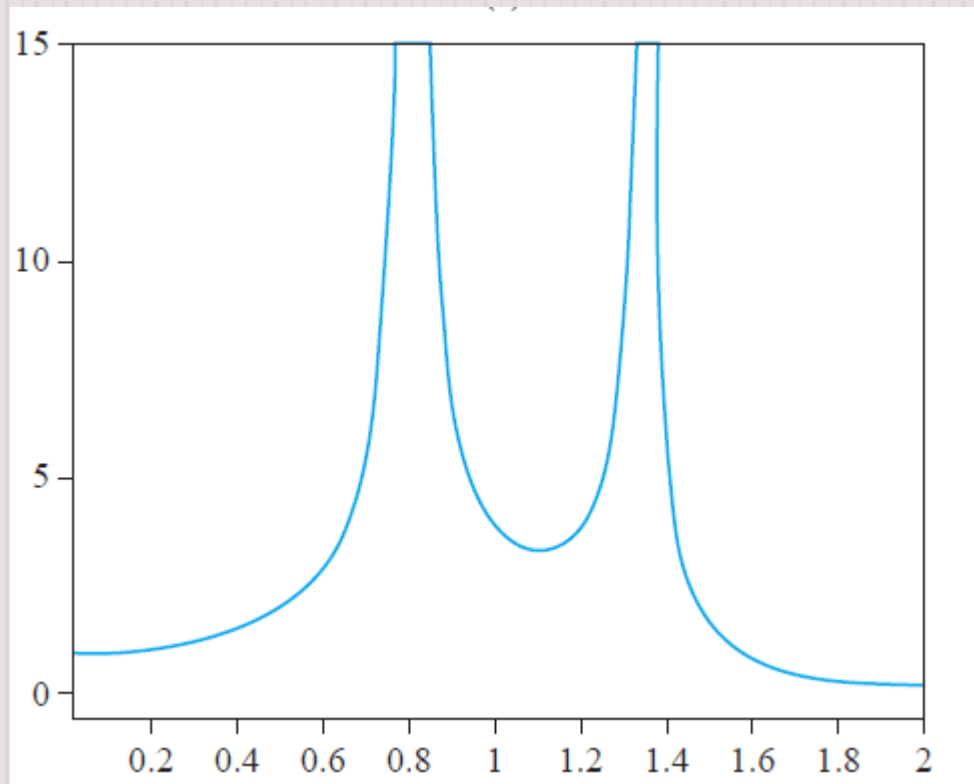
Forced Vibration

$$X_1 = \frac{(k_{22} - m_2 \omega^2) F_1}{(k_{22} - m_2 \omega^2)(k_{11} - m_1 \omega^2) - k_{12} k_{21}}$$



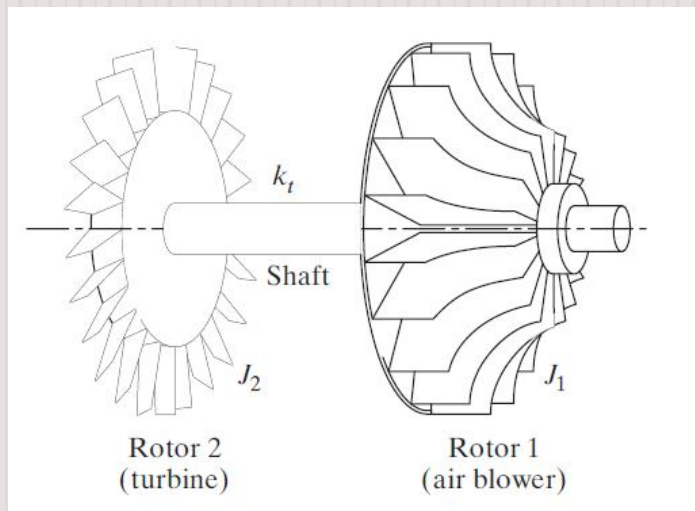
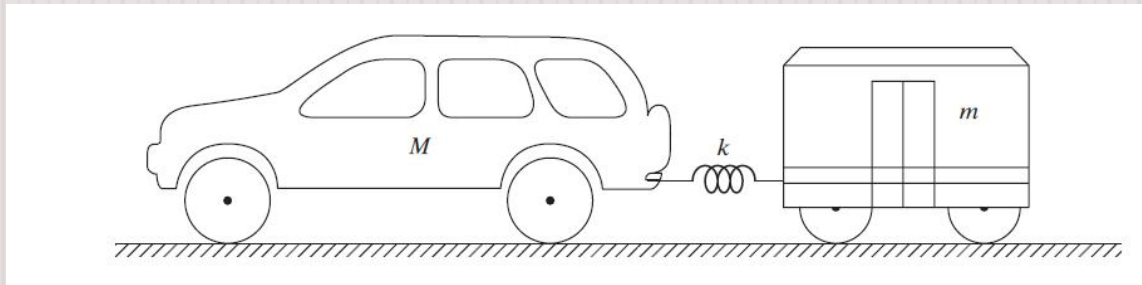
Forced Vibration

$$X_2 = \frac{-k_{21}F_1}{(k_{22} - m_2\omega^2)(k_{11} - m_1\omega^2) - k_{12}k_{21}}$$



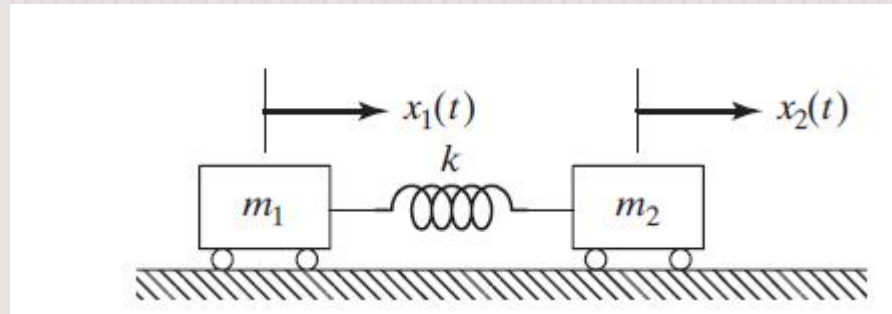
Semidefinite Systems

Semidefinite systems are also known as *unrestrained* or *degenerate systems*. Two examples of such systems are shown in Fig. 5.17. The arrangement in Fig. 5.17(a) may be considered to represent two railway cars of masses m_1 and m_2 with a coupling spring k . The arrangement in Fig. 5.17(c) may be considered to represent two rotors of mass moments of inertia J_1 and J_2 connected by a shaft of torsional stiffness k_t .

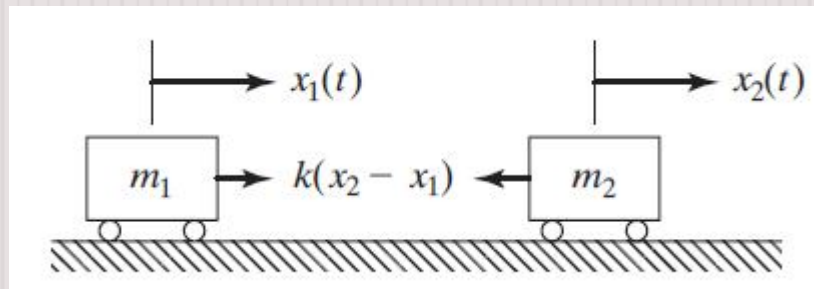


Semidefinite Systems

- Example



- Sol.



$$m_1 \ddot{x}_1 + k(x_1 - x_2) = 0$$

$$m_2 \ddot{x}_2 + k(x_2 - x_1) = 0$$

Semidefinite Systems

$$\begin{aligned}m_1 \ddot{x}_1 + k(x_1 - x_2) &= 0 \\m_2 \ddot{x}_2 + k(x_2 - x_1) &= 0\end{aligned}\tag{5.36}$$

For free vibration, we assume the motion to be harmonic:

$$x_j(t) = X_j \cos(\omega t + \phi_j), \quad j = 1, 2\tag{5.37}$$

Substitution of Eq. (5.37) into Eq. (5.36) gives

$$\begin{aligned}(-m_1 \omega^2 + k)X_1 - kX_2 &= 0 \\-kX_1 + (-m_2 \omega^2 + k)X_2 &= 0\end{aligned}\tag{5.38}$$

By equating the determinant of the coefficients of X_1 and X_2 to zero, we obtain the frequency equation as

$$\omega^2[m_1 m_2 \omega^2 - k(m_1 + m_2)] = 0\tag{5.39}$$

from which the natural frequencies can be obtained:

$$\omega_1 = 0 \text{ and } \omega_2 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}\tag{5.40}$$

Energy Method

Lagrange Equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = Q_i, \quad i = 1, 2, \dots, n$$

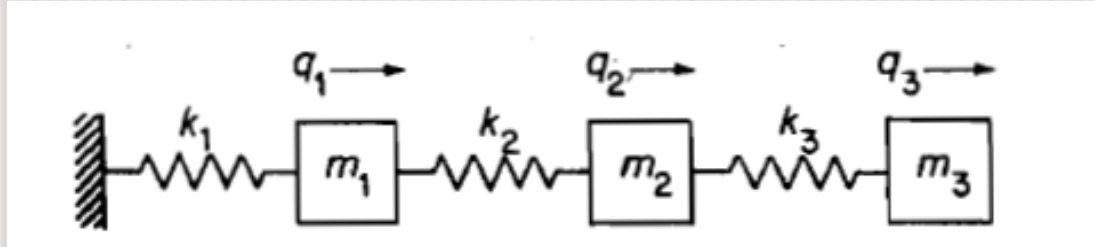
$$L = T - V \quad \text{or} \quad L = T - U$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$

$$\delta W = \sum_{i=1}^N Q_i \delta q_i = Q_1 \delta q_1 + Q_2 \delta q_2 + \dots$$

Energy Method

Example



$$T = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 + \frac{1}{2}m_3\dot{q}_3^2$$

$$U = \frac{1}{2}k_1q_1^2 + \frac{1}{2}k_2(q_2 - q_1)^2 + \frac{1}{2}k_3(q_3 - q_2)^2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$

for $i = 1$,

$$\frac{\partial T}{\partial q_1} = 0 \quad \frac{\partial U}{\partial q_1} = k_1q_1 - k_2(q_2 - q_1)$$

$$\frac{\partial T}{\partial \dot{q}_1} = m_1\dot{q}_1 \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_1} \right) = m_1\ddot{q}_1$$

Energy Method

$$m_1 \ddot{q}_1 + (k_1 + k_2)q_1 - k_2 q_2 = 0$$

$$T = \frac{1}{2}m_1 \dot{q}_1^2 + \frac{1}{2}m_2 \dot{q}_2^2 + \frac{1}{2}m_3 \dot{q}_3^2$$

$$U = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2 (q_2 - q_1)^2 + \frac{1}{2}k_3 (q_3 - q_2)^2$$

For $i = 2$, we have

$$\frac{\partial T}{\partial \dot{q}_2} = m_2 \dot{q}_2 \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_2} \right) = m_2 \ddot{q}_2$$

$$\frac{\partial U}{\partial q_2} = k_2 (q_2 - q_1) - k_3 (q_3 - q_2)$$

$$m_2 \ddot{q}_2 - k_2 q_1 + (k_2 + k_3)q_2 - k_3 q_3 = 0$$

Energy Method

$$T = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 + \frac{1}{2}m_3\dot{q}_3^2$$

$$U = \frac{1}{2}k_1q_1^2 + \frac{1}{2}k_2(q_2 - q_1)^2 + \frac{1}{2}k_3(q_3 - q_2)^2$$

for $i = 3$,

$$\frac{\partial T}{\partial q_3} = m_3\dot{q}_3 \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_3} \right) = m_3\ddot{q}_3$$

$$\frac{\partial U}{\partial q_3} = k_3(q_3 - q_2)$$

$$m_3\ddot{q}_3 - k_3q_2 + k_3q_3 = 0$$

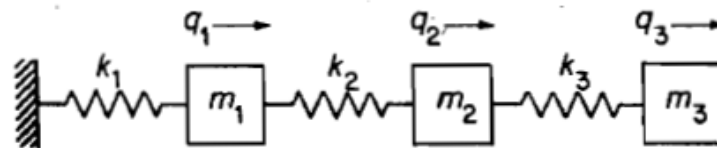
Energy Method

$$m_1 \ddot{q}_1 + (k_1 + k_2)q_1 - k_2 q_2 = 0$$

$$m_2 \ddot{q}_2 - k_2 q_1 + (k_2 + k_3)q_2 - k_3 q_3 = 0$$

$$m_3 \ddot{q}_3 - k_3 q_2 + k_3 q_3 = 0$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$



Energy Method

Example 7.3.2

Using Lagrange's method, set up the equations of motion for the system shown in Fig. 7.3.2.

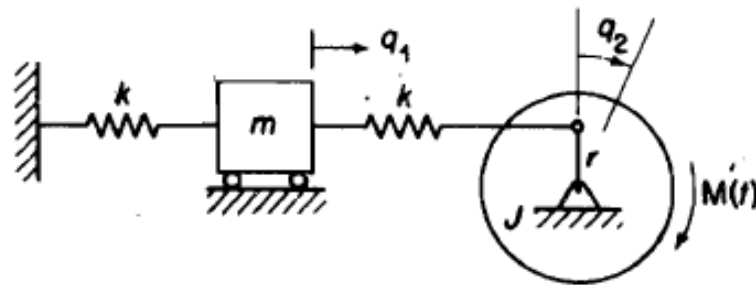


FIGURE 7.3.2.

Solution The kinetic and potential energies are

$$T = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}J\dot{q}_2^2$$

$$U = \frac{1}{2}kq_1^2 + \frac{1}{2}K(rq_2 - q_1)^2$$

and from the work done by the external moment, the generalized force is

$$\delta W = M(t)\delta q_2 \quad \therefore Q_2 = M(t)$$

Energy Method

$$i = 1, \quad \frac{d}{dt}(m\dot{q}_1) - 0 + kq_1 - k(rq_2 - q_1) = 0$$

$$m\ddot{q}_1 + 2kq_1 - krq_2 = 0$$

$$T = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}J\dot{q}_2^2$$

$$U = \frac{1}{2}kq_1^2 + \frac{1}{2}K(rq_2 - q_1)^2$$

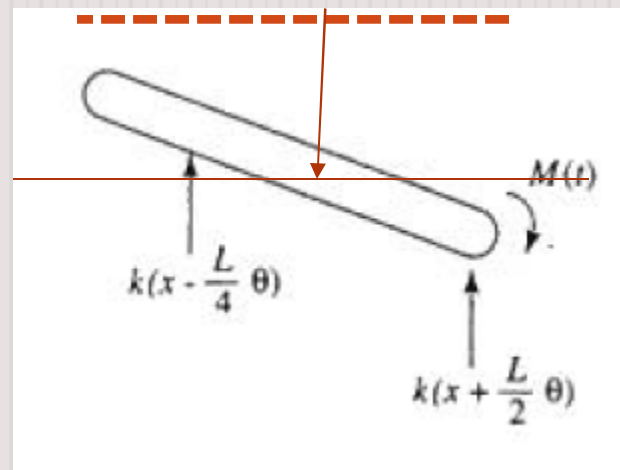
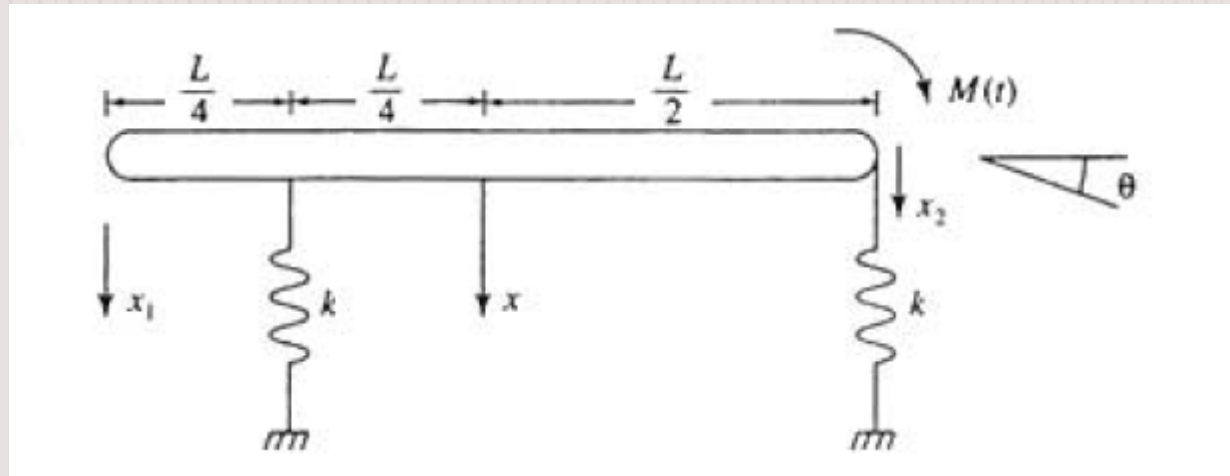
$$i = 2, \quad \frac{d}{dt}(J\dot{q}_2) - 0 + kr(rq_2 - q_1) = M(t)$$

$$J\ddot{q}_2 + kr^2q_2 - krq_1 = M(t)$$

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -kr \\ -kr & kr^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M(t) \end{Bmatrix}$$

Energy Method

- Example



Energy Method

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$$

$$V = \frac{1}{2}k(x - \frac{1}{4}L\theta)^2 + \frac{1}{2}k(x + \frac{1}{2}L\theta)^2$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 - \frac{1}{2}k(x - \frac{1}{4}L\theta)^2 - \frac{1}{2}k(x + \frac{1}{2}L\theta)^2$$

$$\delta W = M(t) \delta \theta$$

Energy Method

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 - \frac{1}{2}k\left(x - \frac{1}{4}L\theta\right)^2 - \frac{1}{2}k\left(x + \frac{1}{2}L\theta\right)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_1$$

$$\frac{d}{dt}(m\dot{x}) + \left[k\left(x - \frac{1}{4}L\theta\right)(1) + k\left(x + \frac{1}{2}L\theta\right)(1)\right] = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_2$$

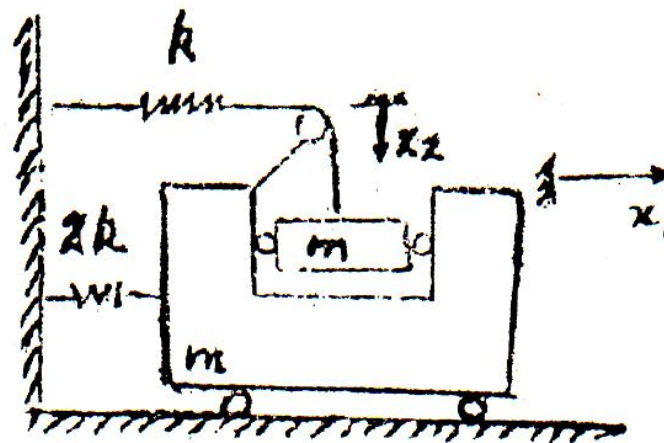
$$\frac{d}{dt}(I\dot{\theta}) + \left[k\left(x - \frac{1}{4}L\theta\right)\left(-\frac{1}{4}L\right) + k\left(x + \frac{1}{2}L\theta\right)\left(\frac{1}{2}L\right)\right] = M(t)$$

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2k & \frac{1}{4}kL \\ \frac{1}{4}kL & \frac{5}{16}kL^2 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ M(t) \end{bmatrix}$$

Energy Method

• Example

برای سیستم نشان داده شده فرکانسها و شکل مودها را تعیین کنید.



معادلات حرکت را با استفاده از معادلات لاگرانژ می نویسیم

$$T = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2} m \dot{x}_1^2$$

$$= m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$U = \frac{1}{2} k (x_1 + x_2)^2 + \frac{1}{2} (2k) x_1^2$$

$$= \frac{3}{2} k x_1^2 + \frac{1}{2} k x_2^2 + k x_1 x_2$$

Energy Method

$$m\ddot{x}_2 + kx_2 + kx_1 = 0$$

$$2m\ddot{x}_1 + 3kx_1 + kx_2 = 0$$

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3k & k \\ k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 3k - 2m\omega^2 & k \\ k & k - m\omega^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \rightarrow (3k - 2m\omega^2)x_1 + kx_2 = 0 \rightarrow \frac{x_1}{x_2} = \frac{k}{2m\omega^2 - 3k} \quad (*)$$

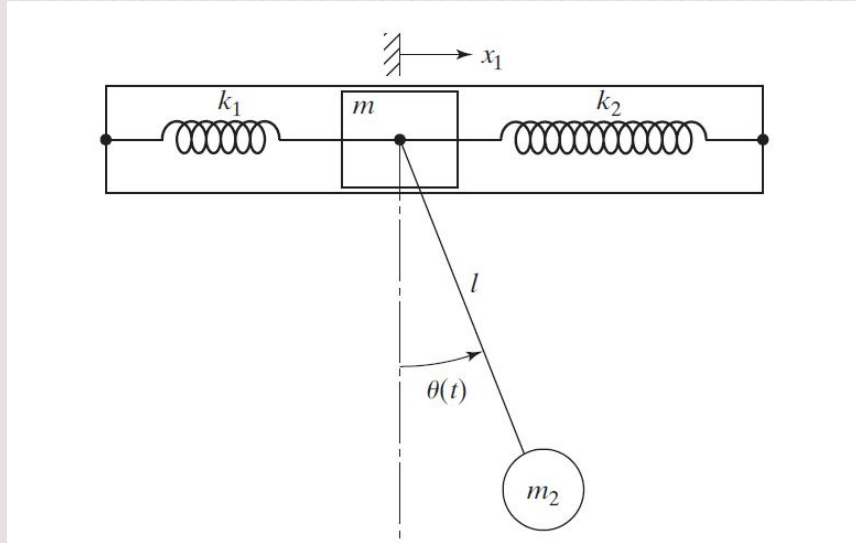
$$\begin{bmatrix} 3k - 2m\omega^2 & k \\ k & k - m\omega^2 \end{bmatrix} = 0 \rightarrow (3k - 2m\omega^2)(k - m\omega^2) - k^2 = 0$$

$$2k^2 + 2m^2\omega^4 - 5mk\omega^2 = 0 \rightarrow \omega_{1,2}^2 = \frac{5mk}{4m^2} \pm \frac{3mk}{4m^2} = \begin{cases} \frac{2k}{m} \\ \frac{k}{2m} \end{cases}$$

$$\begin{matrix} (*) \\ \rightarrow \end{matrix} \frac{x_1}{x_2} \Big|_{\omega=\omega_1} = 1, \quad \frac{x_1}{x_2} \Big|_{\omega=\omega_2} = -0.5 \rightarrow \varphi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \varphi_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$

Energy Method

Example



$$r_1 = x \hat{i} \rightarrow V_1 = \dot{x} \hat{i} \rightarrow |V_1| = \dot{x}$$

$$r_2 = xi + Le^{i\theta} \rightarrow V_2 = \dot{x} \hat{i} + iL\dot{\theta}e^{i\theta} \rightarrow |V_2|^2 = \dot{x}^2 + (L\dot{\theta})^2 + 2\dot{x}(L\dot{\theta})\cos(\theta)$$

$$T = \frac{1}{2}m |V_1|^2 + \frac{1}{2}m_2 |V_2|^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m_2 \left(\dot{x}^2 + (L\dot{\theta})^2 + 2\dot{x}(L\dot{\theta})\cos(\theta) \right)$$

$$U = \frac{1}{2}(K_1 + K_2)x^2 - m_2gL \cos(\theta)$$

Energy Method

$$T = \frac{1}{2} m |\dot{V}_1|^2 + \frac{1}{2} m_2 |\dot{V}_2|^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m_2 \left(\dot{x}^2 + (L \dot{\theta})^2 + 2\dot{x} (L \dot{\theta}) \cos(\theta) \right)$$

$$U = \frac{1}{2} (K_1 + K_2) x^2 - m_2 g L \cos(\theta)$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$

$$q_i = x \rightarrow \frac{d}{dt} \left(m \dot{x} + m_2 \dot{x} + m_2 (L \dot{\theta}) \cos(\theta) \right) - 0 + (K_1 + K_2) x = 0$$

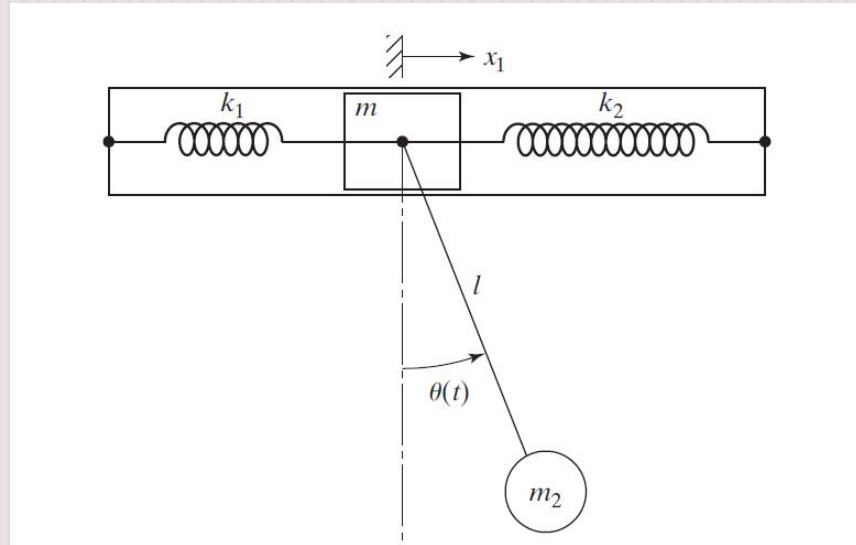
$$\rightarrow (m + m_2) \ddot{x} + m_2 L \cos(\theta) \ddot{\theta} + m_2 L \dot{\theta}^2 \sin(\theta) - 0 + (K_1 + K_2) x = 0$$

$$q_i = \theta \rightarrow \frac{d}{dt} \left(m_2 L^2 \dot{\theta} + m_2 \dot{x} L \cos(\theta) \right) + m_2 \dot{x} (L \dot{\theta}) \sin(\theta) + m_2 g L \sin(\theta) = 0$$

$$\rightarrow m_2 L^2 \ddot{\theta} + m_2 L \cos(\theta) \ddot{x} - m_2 \dot{x} L \dot{\theta} \sin(\theta) + m_2 \dot{x} (L \dot{\theta}) \sin(\theta) + m_2 g L \sin(\theta) = 0$$

$$\rightarrow m_2 L^2 \ddot{\theta} + m_2 L \cos(\theta) \ddot{x} + m_2 g L \sin(\theta) = 0$$

Energy Method



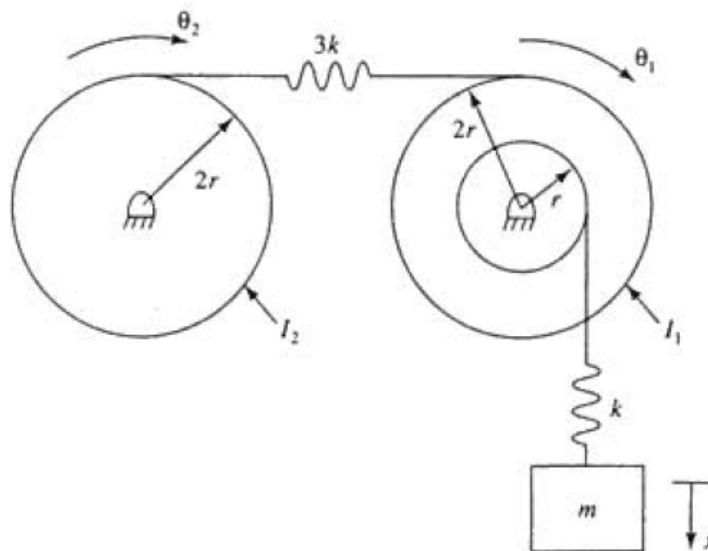
$$\begin{cases} (m + m_2)\ddot{x} + m_2L\ddot{\theta} + (K_1 + K_2)x = 0 \\ m_2L^2\ddot{\theta} + m_2L\ddot{x} + m_2gL\theta = 0 \end{cases}$$

$$\begin{bmatrix} m + m_2 & m_2L \\ m_2L & m_2L^2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (K_1 + K_2) & 0 \\ 0 & m_2gL \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Energy Method

• Example

Use Lagrange's equations to derive the differential equations governing the motion of the system of Fig. 5-5 using x_1 , x_2 , and θ as generalized coordinates. Write the differential equations in matrix form.



The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}m\dot{x}^2$$

The system's potential energy at an arbitrary instant is

$$V = \frac{1}{2}k(x - r\theta_1)^2 + \frac{1}{2}3k(2r\theta_1 - 2r\theta_2)^2$$

Energy Method

The lagrangian is

$$L = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x - r\theta_1)^2 - \frac{1}{2}3k(2r\theta_1 - 2r\theta_2)^2$$

Application of Lagrange's equations leads to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt}(I_1\dot{\theta}_1) + [k(x - r\theta_1)(-r) + 3k(2r\theta_1 - 2r\theta_2)(2r)] = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\frac{d}{dt}(I_2\dot{\theta}_2) + 3k(2r\theta_1 - 2r\theta_2)(-2r) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(m\dot{x}) + k(x - r\theta_1)(1) = 0$$

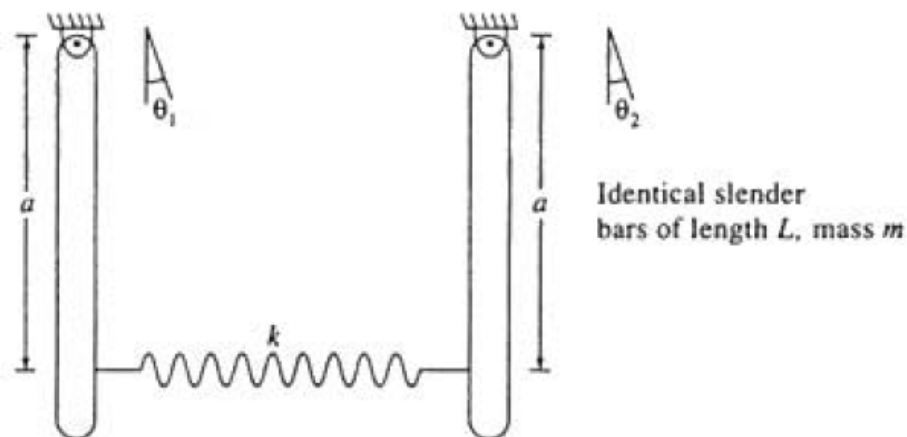
Rearranging and writing in matrix form leads to

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 13kr^2 & -12kr^2 & -kr \\ -12kr^2 & 12kr^2 & 0 \\ -kr & 0 & k \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Energy Method

- Example

Use Lagrange's equations to derive the differential equations governing the motion of the system of Fig. 5-6 using θ_1 and θ_2 as generalized coordinates.



The kinetic energy of the system at an arbitrary instant is

$$T = \frac{1}{2}m\left(\frac{1}{2}L\dot{\theta}_1\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}_1^2 + \frac{1}{2}m\left(\frac{1}{2}L\dot{\theta}_2\right)^2 + \frac{1}{2}\frac{1}{12}mL^2\dot{\theta}_2^2$$

The potential energy of the system at an arbitrary instant is

$$V = -mg\frac{L}{2}\cos\theta_1 - mg\frac{L}{2}\cos\theta_2 + \frac{1}{2}k(a\sin\theta_2 - a\sin\theta_1)^2$$

Energy Method

The lagrangian is

$$L = \frac{1}{2} \frac{1}{3} mL^2 \dot{\theta}_1^2 + \frac{1}{2} \frac{1}{3} mL^2 \dot{\theta}_2^2 + mg \frac{L}{2} \cos \theta_1 + mg \frac{L}{2} \cos \theta_2 \\ - \frac{1}{2} k(a \sin \theta_2 - a \sin \theta_1)^2$$

Application of Lagrange's equations leads to

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt} \left(\frac{1}{3} mL^2 \dot{\theta}_1 \right) + \left[mg \frac{L}{2} \sin \theta_1 + k(a \sin \theta_2 - a \sin \theta_1)(-a \cos \theta_1) \right] = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\frac{d}{dt} \left(\frac{1}{3} mL^2 \dot{\theta}_2 \right) + \left[mg \frac{L}{2} \sin \theta_2 + k(a \sin \theta_2 - a \sin \theta_1)(a \cos \theta_2) \right] = 0$$

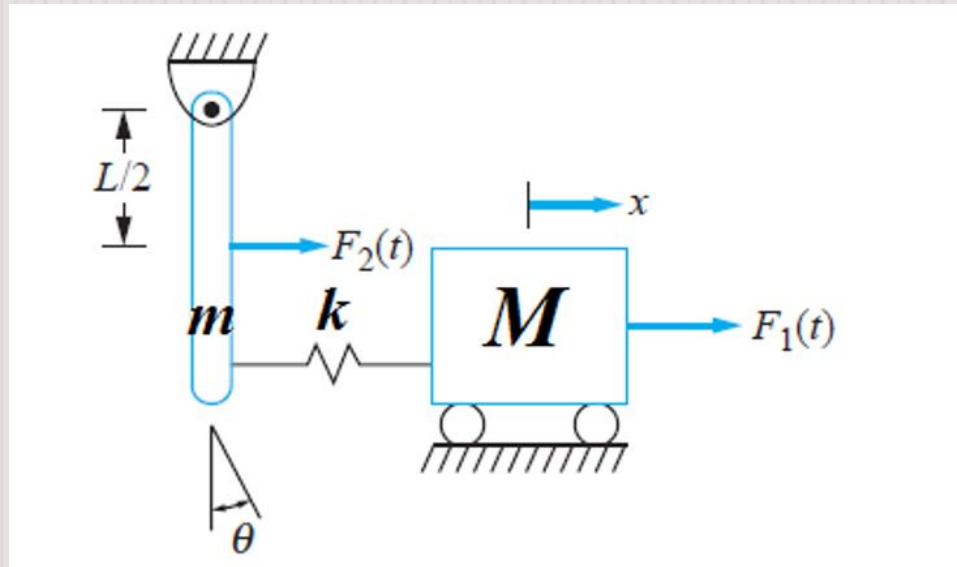
Linearizing and rearranging leads to

$$\frac{1}{3} mL^2 \ddot{\theta}_1 + \left(mg \frac{L}{2} + ka^2 \right) \theta_1 - ka^2 \theta_2 = 0$$

$$\frac{1}{3} mL^2 \ddot{\theta}_2 - ka^2 \theta_1 + \left(mg \frac{L}{2} + ka^2 \right) \theta_2 = 0$$

Energy Method

- Example – Solving by Matlab



$$T = \frac{1}{2} M \dot{\chi}^2 + \frac{1}{2} \left(\frac{1}{3} m L^2 \right) \dot{\theta}^2$$

$$U = \frac{1}{2} k (\chi - L\theta)^2 + mg \left(\frac{L}{2} + \frac{L}{2} (1 - \cos \theta) \right)$$

$$\delta W = F_1 \delta \chi + F_2 \frac{L}{2} \delta \theta$$

Energy Method

$$\ell = T - U = \frac{1}{2} M \dot{\chi}^2 + \frac{1}{2} \left(\frac{1}{3} m L^2 \right) \dot{\theta}^2 - \frac{1}{2} k (\chi - L\theta)^2 - mg \left(\frac{L}{2} + \frac{L}{2} (1 - \cos \theta) \right)$$

$$\frac{d}{dt} \left(\frac{\partial \ell}{\partial \dot{\chi}} \right) - \left(\frac{\partial \ell}{\partial \chi} \right) = Q_1 \rightarrow \frac{d}{dt} (M \dot{\chi}) + k (\chi - L\theta) = F_1 \rightarrow M \ddot{\chi} + k \chi - kL\theta = F_1$$

$$\frac{d}{dt} \left(\frac{\partial \ell}{\partial \dot{\theta}} \right) - \left(\frac{\partial \ell}{\partial \theta} \right) = Q_2 \rightarrow \frac{d}{dt} \left(\frac{1}{3} m L^2 \dot{\theta} \right) - kL (\chi - L\theta) + mg \frac{L}{2} \sin \theta = F_2$$

$$\rightarrow \frac{1}{3} m L^2 \ddot{\theta} + \left(kL^2 + mg \frac{L}{2} \right) \theta - kL \chi = \frac{L}{2} F_2$$

$$\begin{bmatrix} M & 0 \\ 0 & \frac{1}{3} m L^2 \end{bmatrix} \begin{Bmatrix} \ddot{\chi} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & -kL \\ -kL & kL^2 + mg \frac{L}{2} \end{bmatrix} \begin{Bmatrix} \chi \\ \theta \end{Bmatrix} = \begin{Bmatrix} F_1 \\ \frac{L}{2} F_2 \end{Bmatrix}$$

Energy Method

- Solving by Matlab (Parametric)

```
clc;clear;
```

```
syms L m M k w g
```

```
mm=[M 0
```

```
0 m*L*L/3];
```

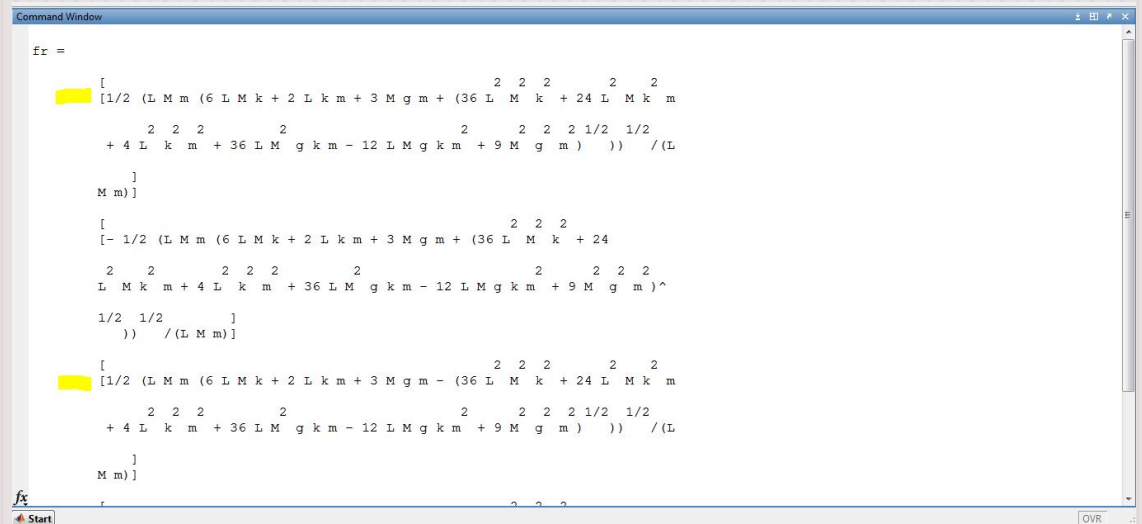
```
kk=[k -k*L
```

```
-k*L k*L*L+m*g*L/2];
```

```
A=[kk-mm*w*w];
```

```
D=simplify(det(A));
```

```
fr=factor(simplify(solve(D,w)))
```



```
Command Window

fr =

[ 1/2 (L M m (6 L M k + 2 L k m + 3 M g m + (36 L M k + 24 L M k m
+ 4 L k m + 36 L M g k m - 12 L M g k m + 9 M g m) ) ) ) / (L
M m) ]

[ - 1/2 (L M m (6 L M k + 2 L k m + 3 M g m + (36 L M k + 24
L M k m + 4 L k m + 36 L M g k m - 12 L M g k m + 9 M g m) ) )
1/2 1/2
) ) / (L M m) ]

[ 1/2 (L M m (6 L M k + 2 L k m + 3 M g m - (36 L M k + 24 L M k m
+ 4 L k m + 36 L M g k m - 12 L M g k m + 9 M g m) ) ) ) / (L
M m) ]
```

Energy Method

- Solving by Matlab (Numeric)

```
clc;clear;
```

```
L=1; m=1; M=2; k=1000; g=9.81;
```

```
mm=[M 0
```

```
0 m*L*L/3];
```

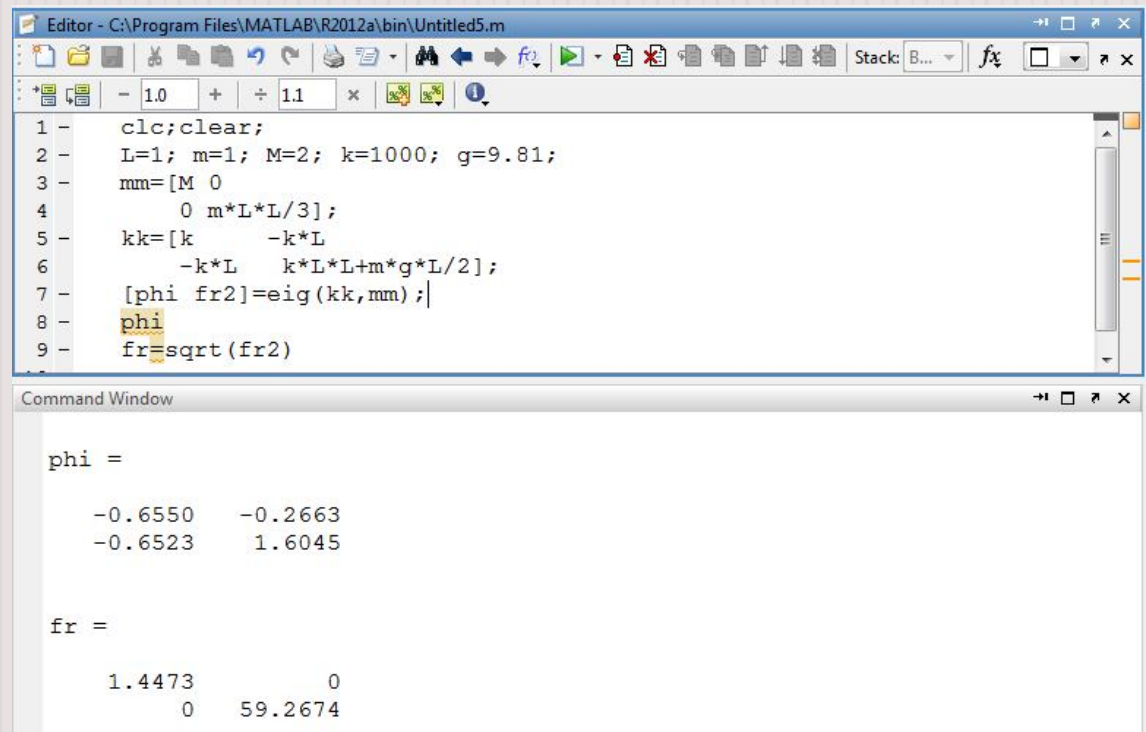
```
kk=[k -k*L
```

```
-k*L k*L*L+m*g*L/2];
```

```
[phi fr2]=eig(kk,mm);
```

```
phi
```

```
fr=sqrt(fr2)
```



The screenshot shows the MATLAB Editor window with the following code:

```
1 - clc;clear;
2 - L=1; m=1; M=2; k=1000; g=9.81;
3 - mm=[M 0
4 -     0 m*L*L/3];
5 - kk=[k -k*L
6 -     -k*L k*L*L+m*g*L/2];
7 - [phi fr2]=eig(kk,mm);
8 - phi
9 - fr=sqrt(fr2)
```

The Command Window displays the results:

```
phi =
    -0.6550    -0.2663
    -0.6523     1.6045

fr =
    1.4473     0
         0    59.2674
```